

Recursive structure in the definitions of gauge-invariant variables for any order perturbations

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Abstract. The construction of gauge-invariant variables for any order perturbations is discussed. Explicit constructions of the gauge-invariant variables for perturbations to 4th order are shown. From these explicit constructions, the recursive structure in the definitions of gauge-invariant variables for any order perturbations is found. Through this recursive structure, the correspondence with the fully non-linear exact perturbations is briefly discussed.

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1. Introduction

Higher-order perturbation theory is one of topical subjects in the recent research on general relativity and have very wide applications: cosmological perturbations [1]; black hole perturbations [2]; and perturbations of stars [3]. However, the “gauge issues” in higher-order perturbations are very delicate in spite of their wide applications. Therefore, it is worthwhile to discuss the higher-order perturbation theory in general relativity from general point of view. Due to this motivation, we have been formulating the higher-order perturbation theory in general relativity through a gauge-invariant manner [4, 5, 6] and applied our formulation to cosmological perturbations [7]. These works are mainly concerning about the second-order perturbations except for Ref. [4]. In this paper, we discuss the “gauge issues” for any order perturbations.

General relativity is a theory based on general covariance and the notion of “gauge” is introduced in the theory due to this general covariance. In particular, in general-relativistic perturbations, *the second kind gauge* appears in perturbations as Sachs pointed out [8]. In general-relativistic perturbation theory, we usually treat one-parameter family of spacetimes $\{(\mathcal{M}_\lambda, Q_\lambda) | \lambda \in [0, 1]\}$ to discuss differences between the background spacetime $(\mathcal{M}_0, Q_0) = (\mathcal{M}_{\lambda=0}, Q_{\lambda=0})$ and the physical spacetime $(\mathcal{M}_{\lambda=1}, Q_{\lambda=1})$. Here, λ is the infinitesimal parameter for perturbations, \mathcal{M}_λ is a

spacetime manifold for each λ , and Q_λ is the collection of the tensor fields on \mathcal{M}_λ . Since each \mathcal{M}_λ is different manifold, we have to introduce the point-identification map $\mathcal{X}_\lambda : \mathcal{M}_0 \mapsto \mathcal{M}_\lambda$ to compare the tensor field on different manifolds. This point-identification is *the gauge choice of the second kind*. Since we have no guiding principle to choose the identification map \mathcal{X}_λ due to the general covariance, we may choose a different point-identification \mathcal{Y}_λ from \mathcal{X}_λ . This degree of freedom of the choice is *the gauge degree of freedom of the second kind*. The *gauge-transformation of the second kind* is a change of this identification map. We note that this second-kind gauge is different notion of the degree of freedom of coordinate choices on a single manifold, which is called *the gauge of the first kind*. Henceforth, we concentrate only on gauge of the second kind and we call this second kind gauge as *gauge* for short.

Once we introduce the gauge choice $\mathcal{X}_\lambda : \mathcal{M}_0 \mapsto \mathcal{M}_\lambda$, we can compare the tensor fields on different manifolds $\{\mathcal{M}_\lambda\}$ and *perturbations* of a tensor field Q_λ are represented by the difference

$$\mathcal{X}_\lambda^* Q_\lambda - Q_0, \quad (1)$$

where \mathcal{X}_λ^* is the pull-back induced by the gauge choice \mathcal{X}_λ and Q_0 is the background value of the variable Q_λ . We note that this representation of perturbations are completely depends on gauge choice \mathcal{X}_λ . If we change the gauge choice from \mathcal{X}_λ to \mathcal{Y}_λ , the pulled-back variable of Q_λ represented by the different representation $\mathcal{Y}_\lambda^* Q_\lambda$. These different representations are related to the gauge-transformation rules as

$$\mathcal{Y}_\lambda^* Q_\lambda = \Phi_\lambda^* \mathcal{X}_\lambda^* Q_\lambda, \quad (2)$$

where

$$\Phi_\lambda := (\mathcal{X}_\lambda)^{-1} \circ \mathcal{Y}_\lambda \quad (3)$$

is a diffeomorphism on \mathcal{M}_0 .

In the perturbative approach, we treat the perturbation $\mathcal{X}_\lambda^* Q_\lambda$ through the Taylor series with respect to the infinitesimal parameter λ as

$$\mathcal{X}_\lambda^* Q_\lambda = \sum_{n=0}^k \frac{\lambda^n}{n!} \mathcal{X}^{(n)} Q + O(\lambda^{k+1}), \quad (4)$$

where $\mathcal{X}^{(k)} Q$ is the representation associated with the gauge choice \mathcal{X}_λ of the k th order perturbation of the variable Q_λ with its background value $\mathcal{X}^{(0)} Q = Q_0$. Similarly, we can have the representation of the perturbation of the variable Q_λ under the gauge choice \mathcal{Y}_λ which is different from \mathcal{X}_λ as mentioned above. Since these different representations are related to the gauge-transformation rule (2), the order-by-order gauge-transformation rule between n th-order perturbations $\mathcal{X}^{(n)} Q$ and $\mathcal{Y}^{(n)} Q$ are given from the Taylor expansion of the gauge-transformation rule (2).

Since Φ_λ is constructed by the product of diffeomorphisms, Φ_λ is not given by an exponential map [4, 7, 9, 10], in general. For this reason, Sonogo and Bruni [10] introduced the notion of a *knight diffeomorphism*. The knight diffeomorphism, which are generated by many generators, includes wider class of diffeomorphisms than exponential

maps which are generated by a single vector field. This knight diffeomorphism is suitable for our order-by-order arguments on the gauge issues of general-relativistic higher-order perturbations. Sonego and Bruni also derived the gauge-transformation rules for any order perturbations.

The purpose of this paper is to point out the recursive structure in the definition of the gauge-invariant variables for the n th-order perturbations. We use the gauge-transformation rules for perturbations derived by Sonego and Bruni. We demonstrate the explicit constructions of gauge-invariant variables to 4th order. From these explicit constructions, we found the recursive structure in the definitions of the gauge-invariant variables for the n th-order perturbations based on algebraic recursion relations (Conjecture 4.1) and the decomposition of the linear metric perturbation into its gauge-invariant and gauge-variant parts (Conjecture 3.1).

The organization of this paper is as follows. In section 2, we review the knight diffeomorphism introduced by Sonego and Bruni [10] and gauge-transformation rules derived them. In section 3, we examine the construction of gauge-invariant variables to 4th-order perturbations. These constructions are based on the conjecture which state that we already know how to construct gauge-invariant variables for linear-order metric perturbation (Conjecture 3.1). In section 4, we discuss the recursive structure in the definitions of gauge-invariant variables for n th-order perturbations. Although this discussion is based on the conjecture for an algebraic identities (Conjecture 4.1), this algebraic identities are confirmed to 4th order perturbation within this paper in section 3. In section 5, we discuss the application of our formulae to cosmological perturbations as an example. The final section (section 6) is devoted to the summary and discussions.

2. Gauge-transformation rules of higher-order perturbations

In this section, we briefly review a representation of diffeomorphism proposed by Sonego and Bruni [10], which called a knight diffeomorphism and the gauge-transformation rules for n th-order perturbations. In gauge-invariant perturbation theories, we may concentrate on the diffeomorphism on the background spacetime \mathcal{M}_0 . However, in this section, we denote the spacetime manifold by \mathcal{M} instead of \mathcal{M}_0 , since our arguments are not restricted to a specific background spacetime \mathcal{M}_0 in perturbation theories.

2.1. Knight diffeomorphism

Let $\phi^{(1)}, \dots, \phi^{(k)}$ be exponential maps on \mathcal{M} which are generated by the vector fields $\xi_{(1)}, \dots, \xi_{(k)}$, respectively. From these exponential maps, we can define a new one-parameter family of diffeomorphisms $\Psi_\lambda^{(k)}$ on \mathcal{M} , whose action is given by

$$\Psi_\lambda^{(k)} := \phi_{\lambda^k/k!}^{(k)} \circ \dots \circ \phi_{\lambda^2/2}^{(2)} \circ \phi_\lambda^{(1)}. \quad (5)$$

$\Psi_\lambda^{(k)}$ displaces a point of \mathcal{M} , a parameter interval λ along the integral curve of $\xi_{(1)}$, then an interval $\lambda^2/2$ along the integral curve of $\xi_{(2)}$, and so on. For this reason, Sonego

and Bruni called $\Psi_\lambda^{(k)}$, with a chess-inspired terminology, a *knight diffeomorphism of rank k* . The vector fields $\xi_{(1)}, \dots, \xi_{(k)}$ are called the generators of $\Psi_\lambda^{(k)}$. The notion of this knight diffeomorphism is useful in perturbation theories in the theories of gravity with general covariance. The reason of this usefulness is in the fact that any C^k one-parameter family Φ_λ of diffeomorphisms can always be approximated by a family of knights diffeomorphism of rank k . Actually, in [10], Sonogo and Bruni showed the following theorem:

Theorem 2.1. *Let \mathcal{D} be an appropriate open set in $\{\lambda\} \times \mathcal{M}$ which includes $\{0\} \times \mathcal{M}$, $\lambda \in \mathbb{R}$, and $\Phi_\lambda : \mathcal{D} \rightarrow \mathcal{M}$ be a C^k one-parameter family of diffeomorphisms. Then, there exists a set of exponential maps $\{\phi^{(1)}, \dots, \phi^{(k)}\}$ on \mathcal{M} such that, up to the order λ^{k+1} , the action of Φ_λ is equivalent to the one of the C^k knight diffeomorphisms*

$$\Phi_\lambda = \Psi_\lambda^{(k)} + O(\lambda^{k+1}) = \phi_{\lambda^k/k!}^{(k)} \circ \dots \circ \phi_{\lambda^2/2!}^{(2)} \circ \phi_\lambda^{(1)} + O(\lambda^{k+1}). \quad (6)$$

If Φ and Ψ are two diffeomorphisms of \mathcal{M} such that $\Phi^*f = \Psi^*f$ for every function f , it follows that $\Phi \equiv \Psi$. In order to show that a family of knight $\Psi_\lambda^{(k)}$ approximates any one-parameter family of diffeomorphisms Φ_λ up to the $(k+1)$ th order, it is sufficient to prove that $\Psi_\lambda^{(k)*}f$ and Φ_λ^*f differ by a function that is $O(\lambda^{k+1})$ for all f . We can always generalize the above approximation property of the action of a knight diffeomorphism $\Psi_\lambda^{(k)*}$ for an arbitrary function to that of the action for an arbitrary tensor field. For this reason, Sonogo and Bruni concentrated on Taylor-expansion of the pull-back $\Psi_\lambda^{(k)*}f = \phi_\lambda^{(1)*} \phi_{\lambda^2/2}^{(2)*} \dots \phi_{\lambda^k/k!}^{(k)*}f$ of a knight diffeomorphism for an arbitrary smooth function f on \mathcal{M} . Then they showed the following proposition:

Proposition 2.1. *Let Φ_λ be a one-parameter family of diffeomorphisms, and T a tensor field such that Φ_λ^*T is of class C^k . Then, Φ_λ^*T can be expanded around $\lambda = 0$ as*

$$\Phi_\lambda^*T = \sum_{l=0}^k \lambda^l \sum_{\{j_i\} \in J_l} \mathcal{C}_l(\{j_i\}) \mathcal{L}_{\xi_{(1)}}^{j_1} \dots \mathcal{L}_{\xi_{(l)}}^{j_l} T + O(\lambda^{k+1}). \quad (7)$$

Here, $J_n := \{\{j_i\} | \forall i \in \mathbb{N}, j_i \in \mathbb{N}, \text{ s.t. } \sum_{i=1}^{\infty} i j_i = n\}$ defines the set of indices over which one has to sum in order to obtain the n th-order term,

$$\mathcal{C}_l(\{j_i\}) := \prod_{i=1}^l \frac{1}{(i!)^{j_i} j_i!}, \quad (8)$$

and $O(\lambda^{k+1})$ is a remainder with $O(\lambda^{k+1})/\lambda^k \rightarrow 0$ in the limit $\lambda \rightarrow 0$.

Here, we note that the expression of the right-hand side of equation (7) is just the form of the Taylor-expansion of the right-hand side of equation (5). From this fact, the proposition 2.1, and the fact that $\Phi \equiv \Psi$ if Φ and Ψ are two diffeomorphisms such that $\Phi^*f = \Psi^*f$ for every function f , we reach to the assertion of Theorem 2.1. Therefore, we may regard that the Taylor-expansion (7) in Proposition 2.1 is the most general expression of the pull-back of diffeomorphism on \mathcal{M} and it is sufficient at least when we concentrate on perturbation theories. We also note that the properties of the set J_n of integers are discussed in Appendix A.

2.2. Gauge-transformation rule for the n th-order perturbations

Through the notion of the knights diffeomorphism in the previous section, we derive the gauge-transformation rules for the n th-order perturbations. As mentioned in section 1, the gauge-transformation rule between the pulled-back variables $\mathcal{Y}_\lambda^* Q_\lambda$ and $\mathcal{X}_\lambda^* Q_\lambda$ is given by (2). In perturbation theories, we always use the Taylor-expansion of these variables as in equation (4). To derive the order-by-order gauge-transformation rule for the n th-order perturbation, we have to know the form of the Taylor-expansion of the pull-back Φ_λ^* of diffeomorphism. Then, we use the general expression (7) of the Taylor expansion of diffeomorphisms in Proposition 2.1 by Sonogo and Bruni. Substituting equations (7) and (4) into equation (2), we obtain the order-by-order expression of the gauge-transformation rules between the perturbative variables $^{(n)}_{\mathcal{X}}Q$ and $^{(n)}_{\mathcal{Y}}Q$ as

$$^{(n)}_{\mathcal{Y}}Q - ^{(n)}_{\mathcal{X}}Q = \sum_{l=1}^n \frac{n!}{(n-l)!} \sum_{\{j_i\} \in J_l} \mathcal{C}_l(\{j_i\}) \mathcal{L}_{\xi_{(1)}}^{j_1} \cdots \mathcal{L}_{\xi_{(l)}}^{j_l} ^{(n-l)}_{\mathcal{X}}Q. \quad (9)$$

The order-by-order gauge-transformation rule (9) gives a complete description of the gauge behavior of perturbations at any order.

3. Definitions of gauge-invariant variables to 4th-order perturbations

Inspecting the gauge-transformation rule (9), we define gauge-invariant variables for metric perturbations and for perturbations of arbitrary tensor fields. Since the definitions of gauge-invariant variables for perturbations of arbitrary tensor fields are trivial if we accomplish the separation of the metric perturbations into their gauge-invariant and gauge-variant parts. Therefore, we may concentrate on the metric perturbations.

First, we consider the metric \bar{g}_{ab} on the physical spacetime $(\mathcal{M}_{\lambda=1}, Q_{\lambda=1})$. We expand the pulled-back metric $\mathcal{X}_\lambda^* \bar{g}_{ab}$ to \mathcal{M}_0 through a gauge choice \mathcal{X}_λ as

$$\mathcal{X}_\lambda^* \bar{g}_{ab} = \sum_{n=0}^k \frac{\lambda^n}{n!} ^{(n)}_{\mathcal{X}}g_{ab} + O(\lambda^{k+1}). \quad (10)$$

where $g_{ab} := ^{(0)}_{\mathcal{X}}g_{ab}$ is the metric on the background spacetime \mathcal{M}_0 . Of course, the expansion (10) of the metric depends entirely on the gauge choice \mathcal{X}_λ . Nevertheless, henceforth, we do not explicitly express the index of the gauge choice \mathcal{X}_λ if there is no possibility of confusion.

In [4], we proposed a procedure to construct gauge-invariant variables for higher-order perturbations. Our starting point to construct gauge-invariant variables was the following conjecture for the linear-metric perturbation $h_{ab} := ^{(1)}g_{ab}$:

Conjecture 3.1. *If there is a symmetric tensor field h_{ab} of the second rank, whose gauge transformation rule is*

$$\mathcal{Y}h_{ab} - \mathcal{X}h_{ab} = \mathcal{L}_\sigma g_{ab}, \quad (11)$$

then there exist a tensor field \mathcal{H}_{ab} and a vector field X^a such that h_{ab} is decomposed as

$$h_{ab} =: \mathcal{H}_{ab} + \mathcal{L}_X g_{ab}, \quad (12)$$

where \mathcal{H}_{ab} and X^a are transformed as

$$\mathcal{Y}\mathcal{H}_{ab} - \mathcal{X}\mathcal{H}_{ab} = 0, \quad \mathcal{Y}X^a - \mathcal{X}X^a = \sigma^a \quad (13)$$

under the gauge transformation (11), respectively.

In this conjecture, \mathcal{H}_{ab} is gauge-invariant and we call \mathcal{H}_{ab} as *gauge-invariant part* of the perturbation h_{ab} . On the other hand, the vector field X^a in equation (16) is gauge dependent, and we call X^a as *gauge-variant part* of the perturbation h_{ab} .

In this paper, we assume Conjecture 3.1. This conjecture is quite important in our scenario of the higher-order gauge-invariant perturbation theory. In [6], we proposed an outline of a proof of Conjecture 3.1. This outline of a proof is almost complete for an arbitrary background metric g_{ab} . However, in this outline, there are missing modes for perturbations, which are called *zero modes* and we also pointed out the physical importance of these zero modes in [6]. Therefore, we have to say that Conjecture 3.1 still a conjecture in our scenario of the higher-order gauge-invariant perturbation theory. If we can take these zero modes into our account in the proof of Conjecture 3.1, we may regard that Conjecture 3.1 is a theorem.

Inspecting the order-by-order gauge-transformation rules (9) and based on Conjecture 3.1, we consider the recursive construction of gauge-invariant variables for higher-order metric perturbations. The proposal of this recursive construction is already given in Sec. 5 of Ref. [4]. In this paper, we try to carry out this proposal through the gauge-transformation rule (9) and show that this proposal is reduced to Conjecture 3.1 and recursive relations of gauge-transformation rules for the gauge-variant variables for metric perturbations (Conjecture 4.1 below).

According to equation (9), the order-by-order gauge-transformation rule for the n th-order metric perturbation ${}^{(n)}_{\mathcal{X}}g_{ab}$ is given by

$${}^{(n)}_{\mathcal{Y}}g_{ab} - {}^{(n)}_{\mathcal{X}}g_{ab} = \sum_{l=1}^n \frac{n!}{(n-l)!} \sum_{\{j_i\} \in J_l} \mathcal{C}_l(\{j_i\}) \mathcal{L}_{\xi_{(1)}}^{j_1} \cdots \mathcal{L}_{\xi_{(l)}}^{j_l} {}^{(n-l)}_{\mathcal{X}}g_{ab}. \quad (14)$$

To define the gauge-invariant variables from this gauge-transformation rule, we reconsider the recursive procedure to find gauge-invariant variables proposed in [4].

3.1. First order

Since we assume Conjecture 3.1 in this paper and the gauge-transformation rule for the first-order metric perturbation is given by

$${}^{(1)}_{\mathcal{Y}}g_{ab} - {}^{(1)}_{\mathcal{X}}g_{ab} = \sum_{l=1}^1 \frac{1!}{(1-1)!} \sum_{\{j_i\} \in J_1} \mathcal{C}_1(\{j_i\}) \mathcal{L}_{\xi_{(1)}}^{j_1} g_{ab} = \mathcal{L}_{\xi_{(1)}} g_{ab}. \quad (15)$$

the first-order metric perturbation ${}^{(1)}g_{ab}$ is decomposed as

$${}^{(1)}g_{ab} =: {}^{(1)}\mathcal{H}_{ab} + \mathcal{L}_{(1)X} g_{ab}, \quad (16)$$

$${}^{(1)}_{\mathcal{Y}}\mathcal{H}_{ab} - {}^{(1)}_{\mathcal{X}}\mathcal{H}_{ab} = 0, \quad {}^{(1)}_{\mathcal{Y}}X^a - {}^{(1)}_{\mathcal{X}}X^a = \xi_{(1)}^a. \quad (17)$$

Through the gauge-variant vector field $^{(1)}X^a$, we can define the gauge-invariant variable $^{(1)}\mathcal{Q}$ of the first-order perturbation for an arbitrary tensor field other than the metric as

$$\begin{aligned} ^{(1)}\mathcal{Q} &:= ^{(1)}Q + \sum_{l=1}^1 \frac{1!}{(1-l)!} \sum_{\{j_i\} \in J_l} \mathcal{C}_l(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} {}^{(1-l)}Q \\ &= ^{(1)}Q + \mathcal{L}_{-(1)X} {}^{(0)}Q. \end{aligned} \quad (18)$$

3.2. Second order

The gauge-transformation rule for the second-order metric perturbation is given from equation (14) as

$${}^{(2)}_y g_{ab} - {}^{(2)}_{\mathcal{X}} g_{ab} = \sum_{l=1}^2 \frac{2!}{(2-l)!} \sum_{\{j_i\} \in J_l} \mathcal{C}_l(\{j_i\}) \mathcal{L}_{\xi(1)}^{j_1} \mathcal{L}_{\xi(2)}^{j_2} {}^{(2-l)}_{\mathcal{X}} g_{ab} \quad (19)$$

$$= 2 \mathcal{L}_{\xi(1)} {}^{(1)}_{\mathcal{X}} g_{ab} + \left\{ \mathcal{L}_{\xi(1)}^2 + \mathcal{L}_{\xi(2)} \right\} g_{ab}. \quad (20)$$

To define the gauge-invariant variables for ${}^{(2)}g_{ab}$, we consider the tensor field defined by

$${}^{(2)}\hat{H}_{ab} := {}^{(2)}g_{ab} + 2 \mathcal{L}_{-(1)X} {}^{(1)}g_{ab} + \mathcal{L}_{-(1)X}^2 g_{ab} \quad (21)$$

$$\begin{aligned} &= {}^{(2)}g_{ab} + \frac{2!}{(2-1)!} \sum_{\{j_i\} \in J_1} \mathcal{C}_1(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} {}^{(1)}g_{ab} \\ &\quad + \frac{2!}{(2-2)!} \sum_{\{j_i\} \in J_2 \setminus {}_2J_0^+} \mathcal{C}_{2-1}(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} g_{ab}, \end{aligned} \quad (22)$$

where the vector field $^{(1)}X^a$ is defined as the gauge-variant part of the first-order metric perturbation ${}^{(1)}g_{ab}$ in equation (16) and ${}_2J_0^+ = \{(j_1, j_2, \dots) = (0, 1, 0, 0, \dots)\}$ is defined in Appendix A. From the expressions (19) and (21), it is easy to show that the gauge-transformation rule

$${}^{(2)}_y \hat{H}_{ab} - {}^{(2)}_{\mathcal{X}} \hat{H}_{ab} = \mathcal{L}_{\sigma(2)} g_{ab}, \quad \sigma_{(2)}^a := \xi_{(2)}^a + \hat{\sigma}_{(2)}^a := \xi_{(2)}^a + [\xi_{(1)}, {}^{(1)}_{\mathcal{X}} X]^a. \quad (23)$$

On the other hand, from the expression (22), we obtain

$$\begin{aligned} &{}^{(2)}_y \hat{H}_{ab} - {}^{(2)}_{\mathcal{X}} \hat{H}_{ab} \\ &= {}^{(2)}_y g_{ab} - {}^{(2)}_{\mathcal{X}} g_{ab} \\ &\quad + \frac{2!}{(2-1)!} \sum_{\{j_i\} \in J_1} \mathcal{C}_1(\{j_i\}) \left(\mathcal{L}_{-(1)X}^{j_1} {}^{(1)}_y g_{ab} - \mathcal{L}_{-(1)X}^{j_1} {}^{(1)}_{\mathcal{X}} g_{ab} \right) \\ &\quad + \frac{2!}{(2-2)!} \sum_{\{j_i\} \in J_2 \setminus {}_2J_0^+} \mathcal{C}_{2-1}(\{j_i\}) \left(\mathcal{L}_{-(1)X}^{j_1} - \mathcal{L}_{-(1)X}^{j_1} \right) g_{ab} \\ &= 2! \sum_{\{j_i\} \in J_1} \mathcal{C}_1(\{j_i\}) \left(\mathcal{L}_{-(1)X}^{j_1} - \mathcal{L}_{-(1)X}^{j_1} + \mathcal{L}_{\xi(1)}^{j_1} \right) {}^{(1)}_{\mathcal{X}} g_{ab} \\ &\quad + 2! \left[\sum_{\{j_i\} \in J_2 \setminus {}_2J_0^+} \mathcal{C}_1(\{j_i\}) \left(\mathcal{L}_{\xi(1)}^{j_1} + \mathcal{L}_{-(1)X}^{j_1} - \mathcal{L}_{-(1)X}^{j_1} \right) \right] \end{aligned}$$

$$\begin{aligned}
 & + \sum_{\{j_i\} \in J_1} \mathcal{C}_1(\{j_i\}) \mathcal{L}_{-\mathcal{Y}^{(1)}X}^{j_1} \sum_{\{k_m\} \in J_1} \mathcal{C}_1(\{k_m\}) \mathcal{L}_{\xi(1)}^{k_1} \Big] g_{ab} \\
 & + \mathcal{L}_{\xi(2)} g_{ab}.
 \end{aligned} \tag{24}$$

Since $J_1 = \{j_1 = 1, j_l = 0 \text{ for } l \geq 2\}$, the gauge-transformation rule for the variable $^{(1)}X^a$ in equation (16) trivially yields

$$\sum_{\{j_i\} \in J_1} \mathcal{C}_1(\{j_i\}) \left(\mathcal{L}_{-\mathcal{Y}^{(1)}X}^{j_1} - \mathcal{L}_{-\mathcal{X}^{(1)}X}^{j_1} + \mathcal{L}_{\xi(1)}^{j_1} \right) = 0. \tag{25}$$

Furthermore, comparing equations (23) and (24), we obtain the identity

$$\begin{aligned}
 & 2! \sum_{\{j_i\} \in J_2 \setminus 2J_0^+} \mathcal{C}_1(\{j_i\}) \left(\mathcal{L}_{\xi(1)}^{j_1} + \mathcal{L}_{-\mathcal{Y}^{(1)}X}^{j_1} - \mathcal{L}_{-\mathcal{X}^{(1)}X}^{j_1} \right) \\
 & + 2! \sum_{\{j_i\} \in J_1} \mathcal{C}_1(\{j_i\}) \mathcal{L}_{-\mathcal{Y}^{(1)}X}^{j_1} \sum_{\{k_m\} \in J_1} \mathcal{C}_1(\{k_m\}) \mathcal{L}_{\xi(1)}^{k_1} \\
 & = \mathcal{L}_{\hat{\sigma}(2)}.
 \end{aligned} \tag{26}$$

Then, we obtain the gauge-transformation rule for the variable $^{(2)}\hat{H}_{ab}$ as the first equation in equation (23).

Since the gauge-transformation rule for the variable $^{(2)}\hat{H}_{ab}$ is given in the first equation in equation (23), applying Conjecture 3.1 to the variable $^{(2)}\hat{H}_{ab}$, we can decompose $^{(2)}\hat{H}_{ab}$ as

$$^{(2)}\hat{H}_{ab} =: ^{(2)}\mathcal{H}_{ab} + \mathcal{L}_{(2)X} g_{ab}, \tag{27}$$

where the gauge-transformation rules $^{(2)}\mathcal{H}_{ab}$ and $^{(2)}X^a$ are given by

$$^{(2)}_{\mathcal{Y}}\mathcal{H}_{ab} - ^{(2)}_{\mathcal{X}}\mathcal{H}_{ab} = 0, \quad ^{(2)}X^a - ^{(2)}_{\mathcal{X}}X^a = \xi_{(2)}^a + \hat{\sigma}_{(2)}^a. \tag{28}$$

Thus, we have decompose the second-order metric perturbation $^{(2)}g_{ab}$ into its gauge-invariant and gauge-variant parts as

$$^{(2)}g_{ab} = ^{(2)}\mathcal{H}_{ab} + 2\mathcal{L}_{(1)X}^{(1)}g_{ab} + (\mathcal{L}_{(2)X} - \mathcal{L}_{(1)X}^2)g_{ab}. \tag{29}$$

The substitution of the second equation in (28) into equation (26), we obtain

$$\begin{aligned}
 & 2! \sum_{\{j_i\} \in J_2 \setminus 2J_0^+} \mathcal{C}_1(\{j_i\}) \left(\mathcal{L}_{\xi(1)}^{j_1} + \mathcal{L}_{-\mathcal{Y}^{(1)}X}^{j_1} - \mathcal{L}_{-\mathcal{X}^{(1)}X}^{j_1} \right) \\
 & + 2! \sum_{\{j_i\} \in J_1} \mathcal{C}_1(\{j_i\}) \mathcal{L}_{-\mathcal{Y}^{(1)}X}^{j_1} \sum_{\{k_m\} \in J_1} \mathcal{C}_1(\{k_m\}) \mathcal{L}_{\xi(1)}^{k_1} \\
 & = -\mathcal{L}_{\xi(2)} - \mathcal{L}_{-\mathcal{Y}^{(2)}X} + \mathcal{L}_{-\mathcal{X}^{(2)}X}.
 \end{aligned} \tag{30}$$

It is easy to see that the identity (30) is expressed as

$$\begin{aligned}
 & \sum_{\{j_i\} \in J_2} \mathcal{C}_2(\{j_i\}) \left(\mathcal{L}_{\xi(1)}^{j_1} \mathcal{L}_{\xi(2)}^{j_2} + \mathcal{L}_{-\mathcal{Y}^{(1)}X}^{j_1} \mathcal{L}_{-\mathcal{Y}^{(2)}X}^{j_2} - \mathcal{L}_{-\mathcal{X}^{(1)}X}^{j_1} \mathcal{L}_{-\mathcal{X}^{(2)}X}^{j_2} \right) \\
 & + \sum_{\{j_i\} \in J_1} \mathcal{C}_1(\{j_i\}) \mathcal{L}_{-\mathcal{Y}^{(1)}X}^{j_1} \sum_{\{k_m\} \in J_1} \mathcal{C}_1(\{k_m\}) \mathcal{L}_{\xi(1)}^{k_1} = 0.
 \end{aligned} \tag{31}$$

As shown in [4], through the gauge-variant variables $^{(2)}X^a$ and $^{(1)}X^a$, we can always define the gauge-invariant variables $^{(2)}Q$ for the second-order perturbation of an arbitrary tensor field other than the metric as

$$\begin{aligned} ^{(2)}Q &:= ^{(2)}Q + \sum_{l=1}^2 \frac{2!}{(2-l)!} \sum_{\{j_i\} \in J_l} \mathcal{C}_l(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} \mathcal{L}_{-(2)X}^{j_l} {}^{(2-l)}Q \\ &= ^{(2)}Q + 2\mathcal{L}_{-(1)X} {}^{(1)}Q + \{\mathcal{L}_{-(2)X} + \mathcal{L}_{-(1)X}^2\} {}^{(0)}Q. \end{aligned} \quad (32)$$

3.3. Third order

The gauge-transformation rule for the third-order metric perturbation is given from equation (14) as

$$\begin{aligned} {}^{(3)}_y g_{ab} - {}^{(3)}_{\mathcal{X}} g_{ab} &= \sum_{l=1}^3 \frac{3!}{(3-l)!} \sum_{\{j_i\} \in J_l} \mathcal{C}_l(\{j_i\}) \mathcal{L}_{\xi(1)}^{j_1} \cdots \mathcal{L}_{\xi(l)}^{j_l} {}^{(3-l)}_{\mathcal{X}} g_{ab} \\ &= 3\mathcal{L}_{\xi(1)} {}^{(2)}_{\mathcal{X}} g_{ab} + 3 \left(\mathcal{L}_{\xi(1)}^2 + \mathcal{L}_{\xi(2)} \right) {}^{(1)}_{\mathcal{X}} g_{ab} \\ &\quad + \left(\mathcal{L}_{\xi(1)}^3 + 3\mathcal{L}_{\xi(1)} \mathcal{L}_{\xi(2)} + \mathcal{L}_{\xi(3)} \right) g_{ab}. \end{aligned} \quad (33)$$

To define the gauge-invariant variables for $^{(2)}g_{ab}$, we consider the tensor field defined by

$$\begin{aligned} {}^{(3)}\hat{H}_{ab} &:= {}^{(3)}g_{ab} + 3\mathcal{L}_{-(1)X} {}^{(2)}g_{ab} + 3 \left(\mathcal{L}_{-(1)X}^2 + \mathcal{L}_{-(2)X} \right) {}^{(1)}g_{ab} \\ &\quad + \left(\mathcal{L}_{-(1)X}^3 + 3\mathcal{L}_{-(1)X} \mathcal{L}_{-(2)X} \right) g_{ab} \end{aligned} \quad (34)$$

$$\begin{aligned} &= {}^{(3)}g_{ab} + \sum_{l=1}^2 \frac{3!}{(3-l)!} \sum_{\{j_i\} \in J_l} \mathcal{C}_l(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(l)X}^{j_l} {}^{(3-l)}g_{ab} \\ &\quad + 3! \sum_{\{j_i\} \in J_3 \setminus {}_3J_0^+} \mathcal{C}_3(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(3)X}^{j_3} g_{ab}. \end{aligned} \quad (35)$$

As shown in [4], directly from the expression (35), we have shown the gauge-transformation rule for the variable $^{(3)}\hat{H}_{ab}$ is given as

$${}^{(3)}_y \hat{H}_{ab} - {}^{(3)}_{\mathcal{X}} \hat{H}_{ab} = \mathcal{L}_{\sigma(3)} g_{ab}, \quad (36)$$

$$\sigma_{(3)}^a := \xi_{(3)}^a + \hat{\sigma}_{(3)}^a, \quad (37)$$

$$\begin{aligned} \hat{\sigma}_{(3)}^a &:= 3[\xi_{(1)}, \xi_{(2)}]^a + 3[\xi_{(1)}, {}^{(2)}_{\mathcal{X}} X]^a + 2[\xi_{(1)}, [\xi_{(1)}, {}^{(1)}_{\mathcal{X}} X]]^a \\ &\quad + [{}^{(1)}_{\mathcal{X}} X, [\xi_{(1)}, {}^{(1)}_{\mathcal{X}} X]]^a. \end{aligned} \quad (38)$$

On the other hand, from the expression (36), the gauge-transformation rule for the variable $^{(3)}\hat{H}_{ab}$ is also given as

$$\begin{aligned} &{}^{(3)}_y \hat{H}_{ab} - {}^{(3)}_{\mathcal{X}} \hat{H}_{ab} \\ &= \frac{3!}{2!} \sum_{\{j_i\} \in J_1} \mathcal{C}_1(\{j_i\}) \left(\mathcal{L}_{-yX}^{j_1} - \mathcal{L}_{-\mathcal{X}X}^{j_1} + \mathcal{L}_{\xi(1)}^{j_1} \right) {}^{(2)}_{\mathcal{X}} g_{ab} \\ &\quad + 3! \left[\sum_{\{j_i\} \in J_2} \mathcal{C}_2(\{j_i\}) \left(\mathcal{L}_{-yX}^{j_1} \mathcal{L}_{-yX}^{j_2} - \mathcal{L}_{-\mathcal{X}X}^{j_1} \mathcal{L}_{-\mathcal{X}X}^{j_2} - \mathcal{L}_{\xi(1)}^{j_1} \mathcal{L}_{\xi(2)}^{j_2} \right) \right. \\ &\quad \left. + 3! \sum_{\{j_i\} \in J_3} \mathcal{C}_3(\{j_i\}) \left(\mathcal{L}_{-yX}^{j_1} \mathcal{L}_{-yX}^{j_2} \mathcal{L}_{-yX}^{j_3} - \mathcal{L}_{-\mathcal{X}X}^{j_1} \mathcal{L}_{-\mathcal{X}X}^{j_2} \mathcal{L}_{-\mathcal{X}X}^{j_3} - \mathcal{L}_{\xi(1)}^{j_1} \mathcal{L}_{\xi(2)}^{j_2} \mathcal{L}_{\xi(3)}^{j_3} \right) \right] g_{ab} \end{aligned}$$

$$\begin{aligned}
 & + \sum_{\{j_i\} \in J_1} \mathcal{C}_1(\{j_i\}) \mathcal{L}_{-\mathcal{Y}X}^{j_1} \sum_{\{k_m\} \in J_1} \mathcal{C}_1(\{k_m\}) \mathcal{L}_{\xi(1)}^{k_1} \Big] \mathcal{X}^{(1)} g_{ab} \\
 & + 3! \left[\sum_{\{j_i\} \in J_3 \setminus \mathcal{J}_0^+} \mathcal{C}_2(\{j_i\}) \left(\mathcal{L}_{\xi(1)}^{j_1} \mathcal{L}_{\xi(2)}^{j_2} + \mathcal{L}_{-\mathcal{Y}X}^{j_1} \mathcal{L}_{-\mathcal{Y}X}^{j_2} - \mathcal{L}_{-\mathcal{X}X}^{j_1} \mathcal{L}_{-\mathcal{X}X}^{j_2} \right) \right. \\
 & + \sum_{\{j_i\} \in J_1} \mathcal{C}_1(\{j_i\}) \mathcal{L}_{-\mathcal{Y}X}^{j_1} \sum_{\{k_m\} \in J_2} \mathcal{C}_2(\{k_m\}) \mathcal{L}_{\xi(1)}^{k_1} \mathcal{L}_{\xi(2)}^{k_2} \\
 & + \sum_{\{j_i\} \in J_2} \mathcal{C}_2(\{j_i\}) \mathcal{L}_{-\mathcal{Y}X}^{j_1} \mathcal{L}_{-\mathcal{Y}X}^{j_2} \sum_{\{k_m\} \in J_1} \mathcal{C}_1(\{k_m\}) \mathcal{L}_{\xi(1)}^{k_1} \Big] g_{ab} \\
 & + \mathcal{L}_{\xi(3)} g_{ab} \tag{40} \\
 = & + 3! \left[\sum_{\{j_i\} \in J_3 \setminus \mathcal{J}_0^+} \mathcal{C}_2(\{j_i\}) \left(\mathcal{L}_{\xi(1)}^{j_1} \mathcal{L}_{\xi(2)}^{j_2} + \mathcal{L}_{-\mathcal{Y}X}^{j_1} \mathcal{L}_{-\mathcal{Y}X}^{j_2} - \mathcal{L}_{-\mathcal{X}X}^{j_1} \mathcal{L}_{-\mathcal{X}X}^{j_2} \right) \right. \\
 & + \sum_{\{j_i\} \in J_1} \mathcal{C}_1(\{j_i\}) \mathcal{L}_{-\mathcal{Y}X}^{j_1} \sum_{\{k_i\} \in J_2} \mathcal{C}_2(\{k_m\}) \mathcal{L}_{\xi(1)}^{k_1} \mathcal{L}_{\xi(2)}^{k_2} \\
 & + \sum_{\{j_i\} \in J_2} \mathcal{C}_2(\{j_i\}) \mathcal{L}_{-\mathcal{Y}X}^{j_1} \mathcal{L}_{-\mathcal{Y}X}^{j_2} \sum_{\{k_m\} \in J_1} \mathcal{C}_1(\{k_m\}) \mathcal{L}_{\xi(1)}^{k_1} \Big] g_{ab} \\
 & + \mathcal{L}_{\xi(3)} g_{ab}. \tag{41}
 \end{aligned}$$

To obtain the expression (40), we used the lower-order gauge-transformation rules (15) and (19) for the metric perturbations. Furthermore, we used the identities (25) and (31) to reach the expression (41).

We note that the gauge-transformation rule (37) with equation (39) for the variable ${}^{(3)}\mathcal{H}_{ab}$ yields that

$$\begin{aligned}
 & 3! \sum_{\{j_i\} \in J_3 \setminus \mathcal{J}_0^+} \mathcal{C}_2(\{j_i\}) \left(\mathcal{L}_{\xi(1)}^{j_1} \mathcal{L}_{\xi(2)}^{j_2} + \mathcal{L}_{-\mathcal{Y}X}^{j_1} \mathcal{L}_{-\mathcal{Y}X}^{j_2} - \mathcal{L}_{-\mathcal{X}X}^{j_1} \mathcal{L}_{-\mathcal{X}X}^{j_2} \right) \\
 & + 3! \sum_{\{j_i\} \in J_1} \mathcal{C}_1(\{j_i\}) \mathcal{L}_{-\mathcal{Y}X}^{j_1} \sum_{\{k_m\} \in J_2} \mathcal{C}_2(\{k_m\}) \mathcal{L}_{\xi(1)}^{k_1} \mathcal{L}_{\xi(2)}^{k_2} \\
 & + 3! \sum_{\{j_i\} \in J_2} \mathcal{C}_2(\{j_i\}) \mathcal{L}_{-\mathcal{Y}X}^{j_1} \mathcal{L}_{-\mathcal{Y}X}^{j_2} \sum_{\{k_m\} \in J_1} \mathcal{C}_1(\{k_m\}) \mathcal{L}_{\xi(1)}^{k_1} \\
 = & \mathcal{L}_{\hat{\sigma}(3)}, \tag{42}
 \end{aligned}$$

since the background metric g_{ab} is arbitrary.

On the other hand, the gauge-transformation rule (37) together with Conjecture 3.1 implies that the variable ${}^{(3)}\hat{H}_{ab}$ is decomposed as

$${}^{(3)}\hat{H}_{ab} =: {}^{(3)}\mathcal{H}_{ab} + \mathcal{L}_{(3)X} g_{ab}, \tag{43}$$

where the gauge-transformation rules ${}^{(3)}\mathcal{H}_{ab}$ and ${}^{(3)}X^a$ are given by

$${}^{(3)}_{\mathcal{Y}}\mathcal{H}_{ab} - {}^{(3)}_{\mathcal{X}}\mathcal{H}_{ab} = 0, \quad {}^{(3)}_{\mathcal{Y}}X^a - {}^{(3)}_{\mathcal{X}}X^a = \xi_{(3)}^a + \hat{\sigma}_{(3)}^a. \tag{44}$$

Thus, we have decompose the third-order metric perturbation $^{(3)}g_{ab}$ into its gauge-invariant and gauge-variant parts as

$$^{(3)}g_{ab} := ^{(3)}\mathcal{H}_{ab} - \sum_{l=1}^3 \frac{3!}{(3-l)!} \sum_{\{j_i\} \in J_l} \mathcal{C}_l(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(l)X}^{j_l} {}^{(3-l)}g_{ab}, \quad (45)$$

$$\begin{aligned} &= ^{(3)}\mathcal{H}_{ab} + 3\mathcal{L}_{(1)X} {}^{(2)}g_{ab} + 3(-\mathcal{L}_{(1)X}^2 + \mathcal{L}_{(2)X}) {}^{(1)}g_{ab} \\ &\quad + (\mathcal{L}_{(1)X}^3 - 3\mathcal{L}_{(1)X}\mathcal{L}_{(2)X} + \mathcal{L}_{(3)X}) g_{ab}. \end{aligned} \quad (46)$$

As shown in [4], through the gauge-variant variables $^{(1)}X^a$, $^{(2)}X^a$, and $^{(3)}X^a$, we can always define the gauge-invariant variables $^{(3)}\mathcal{Q}$ for the third-order perturbation of an arbitrary tensor field other than the metric as

$$^{(3)}\mathcal{Q} = ^{(3)}Q + \sum_{l=1}^3 \frac{3!}{(3-l)!} \sum_{\{j_i\} \in J_l} \mathcal{C}_l(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(l)X}^{j_l} {}^{(3-l)}Q. \quad (47)$$

Substitution of the second equation in (44) into equation (42) leads to the identity

$$\begin{aligned} &3! \sum_{\{j_i\} \in J_3 \setminus 3J_0^+} \mathcal{C}_2(\{j_i\}) \left(\mathcal{L}_{\xi(1)}^{j_1} \mathcal{L}_{\xi(2)}^{j_2} + \mathcal{L}_{-\mathcal{Y}X}^{j_1} \mathcal{L}_{-\mathcal{Y}X}^{j_2} - \mathcal{L}_{-\mathcal{X}X}^{j_1} \mathcal{L}_{-\mathcal{X}X}^{j_2} \right) \\ &+ 3! \sum_{\{j_i\} \in J_1} \mathcal{C}_1(\{j_i\}) \mathcal{L}_{-\mathcal{Y}X}^{j_1} \sum_{\{k_i\} \in J_2} \mathcal{C}_2(\{k_m\}) \mathcal{L}_{\xi(1)}^{k_1} \mathcal{L}_{\xi(2)}^{k_2} \\ &+ 3! \sum_{\{j_i\} \in J_2} \mathcal{C}_2(\{j_i\}) \mathcal{L}_{-\mathcal{Y}X}^{j_1} \mathcal{L}_{-\mathcal{Y}X}^{j_2} \sum_{\{k_m\} \in J_1} \mathcal{C}_1(\{k_m\}) \mathcal{L}_{\xi(1)}^{k_1} \\ &= -\mathcal{L}_{-\mathcal{Y}X} {}^{(3)}g_{ab} + \mathcal{L}_{-\mathcal{X}X} {}^{(3)}g_{ab} - \mathcal{L}_{\xi(3)} {}^{(3)}g_{ab}, \end{aligned} \quad (48)$$

which is equivalent to the identity

$$\begin{aligned} &\sum_{\{j_i\} \in J_3} \mathcal{C}_3(\{j_i\}) \left(\mathcal{L}_{\xi(1)}^{j_1} \mathcal{L}_{\xi(2)}^{j_2} \mathcal{L}_{\xi(3)}^{j_3} + \mathcal{L}_{-\mathcal{Y}X}^{j_1} \mathcal{L}_{-\mathcal{Y}X}^{j_2} \mathcal{L}_{-\mathcal{Y}X}^{j_3} \right. \\ &\quad \left. - \mathcal{L}_{-\mathcal{X}X}^{j_1} \mathcal{L}_{-\mathcal{X}X}^{j_2} \mathcal{L}_{-\mathcal{X}X}^{j_3} \right) \\ &+ \sum_{\{j_i\} \in J_1} \mathcal{C}_1(\{j_i\}) \mathcal{L}_{-\mathcal{Y}X}^{j_1} \sum_{\{k_i\} \in J_2} \mathcal{C}_2(\{k_m\}) \mathcal{L}_{\xi(1)}^{k_1} \mathcal{L}_{\xi(2)}^{k_2} \\ &+ \sum_{\{j_i\} \in J_2} \mathcal{C}_2(\{j_i\}) \mathcal{L}_{-\mathcal{Y}X}^{j_1} \mathcal{L}_{-\mathcal{Y}X}^{j_2} \sum_{\{k_m\} \in J_1} \mathcal{C}_1(\{k_m\}) \mathcal{L}_{\xi(1)}^{k_1} \\ &= 0. \end{aligned} \quad (49)$$

3.4. Fourth order

The gauge-transformation rule for the fourth-order metric perturbation is given from equation (14) as

$$^{(4)}_{\mathcal{Y}}g_{ab} - ^{(4)}_{\mathcal{X}}g_{ab} = \sum_{l=1}^4 \frac{4!}{(4-l)!} \sum_{\{j_i\} \in J_l} \mathcal{C}_l(\{j_i\}) \mathcal{L}_{\xi(1)}^{j_1} \cdots \mathcal{L}_{\xi(l)}^{j_l} {}^{(4-l)}_{\mathcal{X}}g_{ab}. \quad (50)$$

Inspecting this gauge-transformation rule, we define the gauge-invariant and gauge-variant variables for ${}^{(4)}g_{ab}$. To do this, as in the case of the second- and third-order perturbations, we consider the tensor field defined by

$$\begin{aligned} {}^{(4)}\hat{H}_{ab} := & {}^{(4)}g_{ab} + \sum_{l=1}^3 \frac{4!}{(4-l)!} \sum_{\{j_i\} \in J_l} \mathcal{C}_l(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(l)X}^{j_l} {}^{(4-l)}g_{ab} \\ & + 4! \sum_{\{j_i\} \in J_4 \setminus 4J_0^+} \mathcal{C}_3(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(3)X}^{j_3} g_{ab}, \end{aligned} \quad (51)$$

where ${}^{(3)}X^a$, ${}^{(2)}X^a$, and ${}^{(1)}X^a$ are defined previously. Through the identities (25), (31), and (49), the gauge-transformation rule for the variable ${}^{(4)}\hat{H}_{ab}$ is given by

$$\begin{aligned} & {}_{\mathcal{Y}}^{(4)}\hat{H}_{ab} - {}_{\mathcal{X}}^{(4)}\hat{H}_{ab} \\ &= \mathcal{L}_{\xi_{(4)}} g_{ab} \\ &+ 4! \left[\sum_{\{j_l\} \in J_4 \setminus 4J_0^+} \mathcal{C}_3(\{j_i\}) \left(\mathcal{L}_{\xi_{(1)}}^{j_1} \cdots \mathcal{L}_{\xi_{(3)}}^{j_3} + \mathcal{L}_{-(1)\mathcal{Y}X}^{j_1} \cdots \mathcal{L}_{-(3)\mathcal{Y}X}^{j_3} \right. \right. \\ &\quad \left. \left. - \mathcal{L}_{-(1)\mathcal{X}X}^{j_1} \cdots \mathcal{L}_{-(3)\mathcal{X}X}^{j_3} \right) \right. \\ &\quad \left. + \sum_{n=1}^3 \sum_{\{j_l\} \in J_n} \mathcal{C}_3(\{j_i\}) \mathcal{L}_{-(1)\mathcal{Y}X}^{j_1} \cdots \mathcal{L}_{-(3)\mathcal{Y}X}^{j_3} \right. \\ &\quad \left. \times \sum_{\{k_m\} \in J_{4-n}} \mathcal{C}_3(\{k_m\}) \mathcal{L}_{\xi_{(1)}}^{k_1} \cdots \mathcal{L}_{\xi_{(3)}}^{k_3} \right] g_{ab}. \end{aligned} \quad (52)$$

Tedious calculations show that the gauge-transformation rule (52) is given by

$${}_{\mathcal{Y}}^{(4)}\hat{H}_{ab} - {}_{\mathcal{X}}^{(4)}\hat{H}_{ab} = \mathcal{L}_{\sigma_{(4)}} g_{ab}, \quad (53)$$

where $\sigma_{(4)}^a$ is given by

$$\begin{aligned} \sigma_{(4)}^a &= \xi_{(4)}^a + \hat{\sigma}_{(4)}^a, \\ \hat{\sigma}_{(4)}^a &= 4[\xi_{(1)}, \xi_{(3)}]^a + 6[\xi_{(1)}, [\xi_{(1)}, \xi_{(2)}]]^a + 4[\xi_{(1)}, {}^{(3)}X]^a \\ &\quad + 3[\xi_{(2)}, {}^{(2)}X]^a + 6[\xi_{(1)}, [\xi_{(1)}, {}^{(2)}X]]^a + 3[\xi_{(2)}, [\xi_{(1)}, {}^{(1)}X]]^a \\ &\quad + 3[{}^{(2)}X, [\xi_{(1)}, {}^{(1)}X]]^a + 3[\xi_{(1)}, [\xi_{(1)}, [\xi_{(1)}, {}^{(1)}X]]]^a \\ &\quad + 3[\xi_{(1)}, [{}^{(1)}X, [\xi_{(1)}, {}^{(1)}X]]]^a + [{}^{(1)}X, [{}^{(1)}X, [\xi_{(1)}, {}^{(1)}X]]]^a. \end{aligned} \quad (55)$$

Then, we may apply Conjecture 3.1 to the variable ${}^{(4)}\hat{H}_{ab}$, we can decompose ${}^{(4)}\hat{H}_{ab}$ into its gauge-invariant and gauge-variant parts as

$${}^{(4)}\hat{H}_{ab} =: {}^{(4)}\mathcal{H}_{ab} + \mathcal{L}_{(4)X} g_{ab}, \quad (56)$$

where the gauge-transformation rules for the variables ${}^{(4)}\mathcal{H}_{ab}$ and ${}^{(4)}X^a$ is given by

$${}_{\mathcal{Y}}^{(4)}\mathcal{H}_{ab} - {}_{\mathcal{X}}^{(4)}\mathcal{H}_{ab} = 0, \quad {}_{\mathcal{Y}}^{(4)}X^a - {}_{\mathcal{X}}^{(4)}X^a = \sigma_{(4)}^a = \xi_{(4)}^a + \hat{\sigma}_{(4)}^a. \quad (57)$$

Thus, we have decompose the fourth-order metric perturbation ${}^{(4)}g_{ab}$ into its gauge-invariant and gauge-variant parts as

$${}^{(4)}g_{ab} = {}^{(4)}\mathcal{H}_{ab} - \sum_{l=1}^4 \frac{4!}{(4-l)!} \sum_{\{j_i\} \in J_l} \mathcal{C}_l(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(l)X}^{j_l} {}^{(4-l)}g_{ab}. \quad (58)$$

As in the case of the lower-order perturbations, we can always define the gauge-invariant variables ${}^{(4)}\mathcal{Q}$ for the fourth-order perturbation of an arbitrary tensor field other than the metric through the gauge-variant parts ${}^{(1)}X^a$, ${}^{(2)}X^a$, ${}^{(3)}X^a$, and ${}^{(4)}X^a$ of the metric perturbations:

$${}^{(4)}\mathcal{Q} := {}^{(4)}Q + \sum_{l=1}^4 \frac{4!}{(4-l)!} \sum_{\{j_i\} \in J_l} \mathcal{C}_l(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(l)X}^{j_l} {}^{(4-l)}Q. \quad (59)$$

We also note that the gauge-transformation rules (52), (53), and the second equation in (57) implies the identity

$$\begin{aligned} & 4! \sum_{\{j_i\} \in J_4 \setminus \mathcal{A}J_0^+} \mathcal{C}_3(\{j_i\}) \left(\mathcal{L}_{\xi(1)}^{j_1} \cdots \mathcal{L}_{\xi(3)}^{j_3} + \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(3)X}^{j_3} \right. \\ & \quad \left. - \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(3)X}^{j_3} \right) \\ & + 4! \sum_{n=1}^3 \sum_{\{j_i\} \in J_n} \mathcal{C}_3(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(3)X}^{j_3} \\ & \quad \times \sum_{\{k_m\} \in J_{4-n}} \mathcal{C}_3(\{k_m\}) \mathcal{L}_{\xi(1)}^{k_1} \cdots \mathcal{L}_{\xi(3)}^{k_3} \\ & = \mathcal{L}_{\hat{\sigma}(4)}. \end{aligned} \quad (60)$$

Substituting the second equation in (57) into (60), we obtain the identity

$$\begin{aligned} & 4! \sum_{\{j_i\} \in J_4 \setminus \mathcal{A}J_0^+} \mathcal{C}_3(\{j_i\}) \left(\mathcal{L}_{\xi(1)}^{j_1} \cdots \mathcal{L}_{\xi(3)}^{j_3} + \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(3)X}^{j_3} \right. \\ & \quad \left. - \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(3)X}^{j_3} \right) \\ & + 4! \sum_{n=1}^3 \sum_{\{j_i\} \in J_n} \mathcal{C}_3(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(3)X}^{j_3} \\ & \quad \times \sum_{\{k_m\} \in J_{4-n}} \mathcal{C}_3(\{k_m\}) \mathcal{L}_{\xi(1)}^{k_1} \cdots \mathcal{L}_{\xi(3)}^{k_3} \\ & = -\mathcal{L}_{\xi(4)} - \mathcal{L}_{-(4)X} + \mathcal{L}_{-(4)X}. \end{aligned} \quad (61)$$

This identity is also expressed as

$$\begin{aligned} & \sum_{\{j_i\} \in J_4} \mathcal{C}_4(\{j_i\}) \left(\mathcal{L}_{\xi(1)}^{j_1} \cdots \mathcal{L}_{\xi(3)}^{j_4} + \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(3)X}^{j_4} \right. \\ & \quad \left. - \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(4)X}^{j_4} \right) \end{aligned}$$

$$\begin{aligned}
 & + \sum_{n=1}^3 \sum_{\{j_l\} \in J_n} \mathcal{C}_3(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(3)X}^{j_3} \\
 & \quad \times \sum_{\{k_m\} \in J_{4-n}} \mathcal{C}_3(\{k_m\}) \mathcal{L}_{\xi(1)}^{k_1} \cdots \mathcal{L}_{\xi(3)}^{k_3} \\
 & = 0,
 \end{aligned} \tag{62}$$

or, equivalently,

$$\begin{aligned}
 & \sum_{n=1}^4 \sum_{\{j_l\} \in J_n} \mathcal{C}_4(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(4)X}^{j_4} \sum_{\{k_m\} \in J_{4-n}} \mathcal{C}_4(\{k_m\}) \mathcal{L}_{\xi(1)}^{k_1} \cdots \mathcal{L}_{\xi(4)}^{k_4} \\
 & = \sum_{\{j_l\} \in J_4} \mathcal{C}_4(\{j_i\}) \mathcal{L}_{-\mathcal{X}X}^{j_1} \cdots \mathcal{L}_{-\mathcal{X}X}^{j_4}.
 \end{aligned} \tag{63}$$

4. Recursive structure in the definitions of gauge-invariant variables for n th-order perturbations

In the last section, we have shown the construction of gauge-invariant variables to 4th order. From these construction, we easily expect that it can be generalize to n th-order perturbations. In this section, we show the scenario of the generalization of the construction of gauge-invariant variables to n th order which can be expected from the results in the last section.

As noted in section 3, the gauge-transformation rule for the n th-order metric perturbation is given by equation (14). Inspecting this gauge-transformation rule, we construct the gauge-invariant variables for ${}^{(n)}g_{ab}$. Through the construction of gauge-invariant variables for ${}^{(i)}g_{ab}$ ($i = 1, \dots, n-1$), we can also define the vector fields ${}^{(i)}X^a$ ($i = 1, \dots, n-1$) are defined through the construction.

$${}^{(i)}\mathcal{Y}X^a - {}^{(i)}\mathcal{X}X^a = \sigma_{(i)}^a = \xi_{(i)}^a + \hat{\sigma}_{(i)}^a. \tag{64}$$

Furthermore, we can also obtain the $n-1$ identities which are expressed as

$$\begin{aligned}
 & \sum_{p=1}^i \sum_{\{j_l\} \in J_p} \mathcal{C}_i(\{j_l\}) \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(i)X}^{j_i} \sum_{\{k_m\} \in J_{i-p}} \mathcal{C}_i(\{k_m\}) \mathcal{L}_{\xi(1)}^{k_1} \cdots \mathcal{L}_{\xi(i)}^{k_i} \\
 & = \sum_{\{j_l\} \in J_i} \mathcal{C}_i(\{j_l\}) \mathcal{L}_{-\mathcal{X}X}^{j_1} \cdots \mathcal{L}_{-\mathcal{X}X}^{j_i}.
 \end{aligned} \tag{65}$$

To define construct the gauge-invariant variables for the metric perturbation ${}^{(n)}g_{ab}$, as in the cases in the last section, we consider the tensor field defined by

$$\begin{aligned}
 {}^{(n)}\hat{H}_{ab} & := {}^{(n)}g_{ab} + \sum_{l=1}^{n-1} \frac{n!}{(n-l)!} \sum_{\{j_i\} \in J_l} \mathcal{C}_l(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(l)X}^{j_l} {}^{(n-l)}g_{ab} \\
 & \quad + n! \sum_{\{j_i\} \in J_n \setminus {}_nJ_0^+} \mathcal{C}_{n-1}(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(n-1)X}^{j_{n-1}} g_{ab}.
 \end{aligned} \tag{66}$$

Using the order-by-order identities (65), the gauge-transformation rule is given by

$$\begin{aligned}
 & {}^{(n)}\hat{H}_{ab} - {}^{(n)}\hat{H}_{ab} \\
 &= \mathcal{L}_{\xi_{(n)}} g_{ab} \\
 &+ n! \left[\sum_{\{j_l\} \in J_n \setminus nJ_0^+} \mathcal{C}_{n-1}(\{j_l\}) \left(\mathcal{L}_{\xi_{(1)}}^{j_1} \cdots \mathcal{L}_{\xi_{(n-1)}}^{j_{n-1}} + \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(n-1)X}^{j_{n-1}} \right. \right. \\
 &\quad \left. \left. - \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(n-1)X}^{j_{n-1}} \right) \right. \\
 &\quad + \sum_{i=1}^{n-1} \sum_{\{j_l\} \in J_i} \mathcal{C}_{n-1}(\{j_l\}) \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(i)X}^{j_i} \\
 &\quad \left. \times \sum_{\{k_m\} \in J_{n-i}} \mathcal{C}_{n-1}(\{k_m\}) \mathcal{L}_{\xi_{(1)}}^{k_1} \cdots \mathcal{L}_{\xi_{(n-1)}}^{k_{n-1}} \right] g_{ab}. \quad (67)
 \end{aligned}$$

From the analyses in the last section, we can expect that the following conjecture is reasonable.

Conjecture 4.1. *There exists a vector field $\hat{\sigma}_{(n)}^a$ such that*

$$\begin{aligned}
 & n! \sum_{\{j_l\} \in J_n \setminus nJ_0^+} \mathcal{C}_{n-1}(\{j_l\}) \left(\mathcal{L}_{\xi_{(1)}}^{j_1} \cdots \mathcal{L}_{\xi_{(n-1)}}^{j_{n-1}} + \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(n-1)X}^{j_{n-1}} \right. \\
 &\quad \left. - \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(n-1)X}^{j_{n-1}} \right) \\
 &+ n! \sum_{i=1}^{n-1} \sum_{\{j_l\} \in J_i} \mathcal{C}_{n-1}(\{j_l\}) \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(i)X}^{j_i} \\
 &\quad \times \sum_{\{k_m\} \in J_{n-i}} \mathcal{C}_{n-1}(\{k_m\}) \mathcal{L}_{\xi_{(1)}}^{k_1} \cdots \mathcal{L}_{\xi_{(n-1)}}^{k_{n-1}} \\
 &= \mathcal{L}_{\hat{\sigma}_{(n)}}. \quad (68)
 \end{aligned}$$

To derive the explicit form of $\hat{\sigma}_{(n)}$, tough algebraic calculations are necessary. Although we do not going to the details of the proof of this conjecture, we expect that this identity should be proved, recursively, and there will be no difficulty except for tough algebraic calculations. Actually, in the last section, we have confirmed this conjecture to 4th order and it is reasonable to regard that Conjecture 4.1 is hold.

If Conjecture 4.1 is hold, the gauge-transformation rule for the variable ${}^{(n)}H_{ab}$ is given by

$${}^{(n)}\hat{H}_{ab} - {}^{(n)}\hat{H}_{ab} = \mathcal{L}_{\sigma_{(n)}} g_{ab}, \quad \sigma_{(n)}^a := \xi_{(n)}^a + \hat{\sigma}_{(n)}^a. \quad (69)$$

This is the same form as the gauge-transformation rule for the linear-order metric perturbation. Then, we may apply Conjecture 3.1 for the variable ${}^{(n)}\hat{H}_{ab}$. This implies that the variable ${}^{(n)}\hat{H}_{ab}$ is decomposed as

$${}^{(n)}\hat{H}_{ab} = {}^{(n)}\mathcal{H}_{ab} + \mathcal{L}_{(n)X} g_{ab}, \quad (70)$$

$${}^{(n)}\mathcal{H}_{ab} - {}^{(n)}\mathcal{H}_{ab} = 0, \quad {}^{(n)}X^a - {}^{(n)}X^a = \sigma_{(n)}^a = \xi_{(n)}^a + \hat{\sigma}_{(n)}^a. \quad (71)$$

Thus, we have gauge-invariant variables $^{(n)}\mathcal{H}_{ab}$ for the n th-order metric perturbation. This implies that the original n th-order metric perturbation $^{(n)}g_{ab}$

$$\begin{aligned}
 ^{(n)}g_{ab} &= ^{(n)}\mathcal{H}_{ab} - \mathcal{L}_{-(n)X}g_{ab} \\
 &\quad - \sum_{l=1}^{n-1} \frac{n!}{(n-l)!} \sum_{\{j_i\} \in J_l} \mathcal{C}_l(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(l)X}^{j_l} ^{(n-l)}g_{ab} \\
 &\quad - n! \sum_{\{j_i\} \in J_n \setminus nJ_0^+} \mathcal{C}_{n-1}(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(n-1)X}^{j_{n-1}} g_{ab} \\
 &= ^{(n)}\mathcal{H}_{ab} - \sum_{l=1}^n \frac{n!}{(n-l)!} \sum_{\{j_i\} \in J_l} \mathcal{C}_l(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(l)X}^{j_l} ^{(n-l)}g_{ab}. \quad (72)
 \end{aligned}$$

This indicate that the n th-order metric perturbation $^{(n)}g_{ab}$ is decomposed as its gauge-invariant, and gauge-variant parts. Through the gauge-variant variables $^{(i)}X^a$ ($i = 1, \dots, n$), we can also define the gauge-invariant variable $^{(n)}\mathcal{Q}$ for the n th-order perturbation $^{(n)}Q$ of any tensor field Q is also defined as

$$^{(n)}\mathcal{Q} := ^{(n)}Q + \sum_{l=1}^n \frac{n!}{(n-l)!} \sum_{\{j_i\} \in J_l} \mathcal{C}_l(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(l)X}^{j_l} ^{(n-l)}Q. \quad (73)$$

Furthermore, Conjecture 4.1 leads the identity which corresponds to (25), (31), (49), and (63). Substituting the second equation in (71) into equation (68), we obtain

$$\begin{aligned}
 n! \sum_{\{j_l\} \in J_n \setminus nJ_0^+} \mathcal{C}_{n-1}(\{j_l\}) &\left(\mathcal{L}_{\xi(1)}^{j_1} \cdots \mathcal{L}_{\xi(n-1)}^{j_{n-1}} + \mathcal{L}_{-\mathcal{Y}X}^{j_1} \cdots \mathcal{L}_{-(n-1)\mathcal{Y}X}^{j_{n-1}} \right. \\
 &\quad \left. - \mathcal{L}_{-\mathcal{X}X}^{j_1} \cdots \mathcal{L}_{-(n-1)\mathcal{X}X}^{j_{n-1}} \right) \\
 &+ n! \sum_{i=1}^{n-1} \sum_{\{j_l\} \in J_i} \mathcal{C}_{n-1}(\{j_l\}) \mathcal{L}_{-\mathcal{Y}X}^{j_1} \cdots \mathcal{L}_{-(n-1)\mathcal{Y}X}^{j_{n-1}} \\
 &\quad \times \sum_{\{k_m\} \in J_{n-i}} \mathcal{C}_{n-1}(\{k_m\}) \mathcal{L}_{\xi(1)}^{k_1} \cdots \mathcal{L}_{\xi(n-1)}^{k_{n-1}} \\
 &= -\mathcal{L}_{\xi(n)} - \mathcal{L}_{-\mathcal{Y}X}^{(n)} + \mathcal{L}_{-\mathcal{X}X}^{(n)}. \quad (74)
 \end{aligned}$$

This is equivalent to

$$\begin{aligned}
 &\sum_{i=1}^n \sum_{\{j_l\} \in J_i} \mathcal{C}_n(\{j_l\}) \mathcal{L}_{-\mathcal{Y}X}^{j_1} \cdots \mathcal{L}_{-(n)X}^{j_n} \sum_{\{k_m\} \in J_{n-i}} \mathcal{C}_n(\{k_m\}) \mathcal{L}_{\xi(1)}^{k_1} \cdots \mathcal{L}_{\xi(n)}^{k_n} \\
 &= \sum_{\{j_l\} \in J_n} \mathcal{C}_n(\{j_l\}) \mathcal{L}_{-\mathcal{X}X}^{j_1} \cdots \mathcal{L}_{-(n)X}^{j_n}. \quad (75)
 \end{aligned}$$

This identity corresponds to the $i = n$ version of identities (65) and is used when we derive the gauge-transformation rules of perturbations higher than n th.

5. Example: Cosmological Perturbations

Here, we consider the application of our formulae derived in the last section to a specific background spacetime as an example. The example discussed here is the cosmological perturbation whose background metric is given by

$$g_{ab} = a^2(\eta) \left(-(d\eta)_a (d\eta)_b + \gamma_{pq} (dx^p)_a (dx^q)_b \right), \quad (76)$$

where $a = a(\eta)$ is the scale factor, γ_{pq} is the metric on the maximally symmetric 3-space with curvature constant K , and the indices p, q, r, \dots for the spatial components run from 1 to 3. In this section, we concentrate only on the metric perturbations.

We have to note that even in the case of this cosmological perturbations, there is the “zero-mode problem” which is mentioned in Sec. 3. In this section, we ignore these zero-modes and assume Conjecture 3.1, for simplicity, because we have not yet resolved the “zero-mode problem” systematically as mentioned in Sec. 3.

On the background spacetime with the metric (76), we consider the metric perturbation $^{(1)}g_{ab}$ and we apply the York decomposition [11]:

$$\begin{aligned} ^{(1)}g_{ab} = & ^{(1)}h_{\eta\eta} (d\eta)_a (d\eta)_b + 2 \left(D_p ^{(1)}h_{(V)L} + ^{(1)}h_{(V)p} \right) (d\eta)_a (dx^p)_b \\ & + a^2 \left\{ ^{(1)}h_{(L)} \gamma_{pq} + \left(D_p D_q - \frac{1}{3} \gamma_{pq} \Delta \right) ^{(1)}h_{(TL)} \right. \\ & \left. + 2 D_p ^{(1)}h_{(TV)q} + ^{(1)}h_{(TT)pq} \right\} (dx^p)_a (dx^q)_b, \end{aligned} \quad (77)$$

where $\Delta := \gamma^{pq} D_p D_q$ and D_p is the covariant derivative associated with the metric γ_{pq} . Here, $^{(1)}h_{(V)p}$, $^{(1)}h_{(TV)p}$, and $^{(1)}h_{(TT)pq}$ satisfy the properties $D^p ^{(1)}h_{(V)p} = D^p ^{(1)}h_{(TV)p} = 0$, $^{(1)}h_{(TT)pq} = ^{(1)}h_{(TT)qp}$, $^{(1)}h_{(TT)p}{}^p := \gamma^{pq} ^{(1)}h_{(TT)pq} = 0$, and $D^p ^{(1)}h_{(TT)pq} = 0$.

The gauge-transformation rules for the variables $\{^{(1)}h_{\eta\eta}, ^{(1)}h_{(V)L}, ^{(1)}h_{(V)p}, ^{(1)}h_{(L)}, ^{(1)}h_{(TL)}, ^{(1)}h_{(TV)q}, ^{(1)}h_{(TT)pq}\}$ are derived from (15). Inspecting these gauge-transformation rules, we define the gauge-variant part $^{(1)}X_a$ in (16):

$$\begin{aligned} ^{(1)}X_a := & \left(^{(1)}h_{(V)L} - \frac{1}{2} a^2 \partial_\eta ^{(1)}h_{(TL)} \right) (d\eta)_a \\ & + a^2 \left(^{(1)}h_{(TV)p} + \frac{1}{2} D_p ^{(1)}h_{(TL)} \right) (dx^p)_a. \end{aligned} \quad (78)$$

We can easily check this vector field $^{(1)}X_a$ satisfies (17). Subtracting gauge-variant part $\mathcal{L}_{^{(1)}X} g_{ab}$ from $^{(1)}g_{ab}$, we have the gauge-invariant part $^{(1)}\mathcal{H}_{ab}$ in (16):

$$\begin{aligned} ^{(1)}\mathcal{H}_{ab} = & a^2 \left\{ -2 ^{(1)}\Phi (d\eta)_a (d\eta)_b + 2 ^{(1)}\nu_p (d\eta)_a (dx^p)_b \right. \\ & \left. + \left(-2 ^{(1)}\Psi \gamma_{pq} + ^{(1)}\chi_{pq} \right) (dx^p)_a (dx^q)_b \right\}, \end{aligned} \quad (79)$$

where the properties $D^p ^{(1)}\nu_p := \gamma^{pq} D_p ^{(1)}\nu_q = 0$, $^{(1)}\chi_p{}^p := \gamma^{pq} ^{(1)}\chi_{pq} := 0$, and $D^p ^{(1)}\chi_{qp} = 0$ are satisfied.

We have to emphasize that, as shown in Refs. [7], the one to one correspondence between the sets of variables $\{^{(1)}g_{\eta\eta}, ^{(1)}g_{\eta p}, ^{(1)}g_{pq}\}$ and $\{^{(1)}h_{\eta\eta}, ^{(1)}h_{(V)L}, ^{(1)}h_{(V)p}, ^{(1)}h_{(L)}, ^{(1)}h_{(TL)}, ^{(1)}h_{(TV)q}, ^{(1)}h_{(TT)pq}\}$ is guaranteed by the existence of the Green functions Δ^{-1} , $(\Delta + 2K)^{-1}$, and $(\Delta + 3K)^{-1}$. In other words, in the decomposition (77), some

perturbative modes of the metric perturbations which belongs to the kernel of the operator Δ , $(\Delta + 2K)$, and $(\Delta + 3K)$ are excluded from our consideration. For example, homogeneous modes belong to the kernel of the operator Δ and are excluded from our consideration. If we have to treat these modes, separate treatments are necessary. This is the “zero-mode problem” in the cosmological perturbations, which was pointed out in Refs. [6].

To define gauge-invariant variables for n th-order metric perturbation, we apply the York decomposition (77) not to the variable $^{(n)}g_{ab}$ but to the variable $^{(n)}\hat{H}_{ab}$ defined by (66):

$$\begin{aligned} ^{(n)}\hat{H}_{ab} = & ^{(n)}h_{\eta\eta}(d\eta)_a(d\eta)_b + 2 \left(D_p ^{(n)}h_{(VL)} + ^{(n)}h_{(V)p} \right) (d\eta)_{(a}(dx^p)_{b)} \\ & + a^2 \left\{ ^{(n)}h_{(L)}\gamma_{pq} + \left(D_p D_q - \frac{1}{3}\gamma_{pq}\Delta \right) ^{(n)}h_{(TL)} \right. \\ & \left. + 2D_{(p} ^{(n)}h_{(TV)q)} + ^{(n)}h_{(TT)pq} \right\} (dx^p)_a(dx^q)_b. \end{aligned} \quad (80)$$

Since the gauge-transformation rule (69) for the variable $^{(n)}\hat{H}_{ab}$ has the same form as the gauge-transformation rule (11), we can define the gauge-variant parts of $^{(n)}\hat{H}_{ab}$ as

$$\begin{aligned} ^{(n)}X_a := & \left(^{(n)}h_{(VL)} - \frac{1}{2}a^2\partial_\eta ^{(n)}h_{(TL)} \right) (d\eta)_a \\ & + a^2 \left(^{(n)}h_{(TV)p} + \frac{1}{2}D_p ^{(n)}h_{(TL)} \right) (dx^p)_a \end{aligned} \quad (81)$$

through the same procedure as the linear case and we can also define the gauge-invariant part $^{(n)}\mathcal{H}_{ab}$ by

$$\begin{aligned} ^{(n)}\mathcal{H}_{ab} = & a^2 \left\{ -2^{(n)}\Phi(d\eta)_a(d\eta)_b + 2^{(n)}\nu_p(d\eta)_{(a}(dx^p)_{b)} \right. \\ & \left. + (-2^{(n)}\Psi\gamma_{pq} + ^{(n)}\chi_{pq}) (dx^p)_a(dx^q)_b \right\}, \end{aligned} \quad (82)$$

where the properties $D^p{}^{(n)}\nu_p := \gamma^{pq}D_p ^{(n)}\nu_q = 0$, $^{(n)}\chi_p{}^p := \gamma^{pq}{}^{(n)}\chi_{pq} := 0$, and $D^p{}^{(n)}\chi_{qp} = 0$ are satisfied.

As noted in Refs. [7], the definitions of gauge-invariant variables are not unique. Therefore, we may choose the different choice of gauge-invariant variables for each order metric perturbations through the different choice of $^{(n)}X_a$. The above choice corresponds to the longitudinal gauge in linear cosmological perturbations.

6. Summary and Discussions

In this paper, we discussed the recursive structure in the construction of gauge-invariant variables for any-order perturbations. As gauge-transformation rules for the higher-order perturbations, we applied the knight diffeomorphism introduced by Sonego and Bruni [10]. This diffeomorphism is regarded as general diffeomorphism in the order-by-order treatment of perturbations. Based on the gauge-transformation rules for higher-order perturbations derived by Sonego and Bruni [10], we proposed the procedure to construct gauge-invariant variables to third order in [4]. Based on this procedure, in this paper, we consider the explicit and systematic construction of gauge-invariant variables

for more higher-order perturbations. As a result, we found that the recursive structure in the construction of gauge-invariant variables.

Although we do not prove Conjecture 4.1 within this paper, we have confirmed this conjecture to 4th order. Therefore, it is reasonable to regard that the algebraic relation (68) is hold. Then, the gauge-transformation rule for the variable ${}^{(n)}\hat{H}_{ab}$ defined by equation (66) is given as equation (69). This indicates that we may apply Conjecture 3.1 to the variable ${}^{(n)}\hat{H}_{ab}$ and we can decompose the metric perturbation ${}^{(n)}g_{ab}$ of n th order into its gauge-invariant part ${}^{(n)}\mathcal{H}_{ab}$ and the gauge-variant part ${}^{(n)}X^a$. The gauge-transformation rule of the gauge-variant part ${}^{(n)}X^a$ leads the identity (65) with $i = n$. The identities (65) with $i = 1, \dots, n$ is used when we derived the gauge-transformation rule for the variable ${}^{(n+1)}\hat{H}_{ab}$ which is given by equation (67) with the replacement $n \rightarrow n + 1$. Through Conjecture 4.1 with the replacement $n \rightarrow n + 1$, the gauge-transformation rule for the variable ${}^{(n+1)}\hat{H}_{ab}$ is also given in the form (69) with the replacement $n \rightarrow n + 1$. Thus, we can recursively construct gauge-invariant variables for any order perturbations through Conjectures 3.1 and 4.1. In this paper, we have confirmed this recursive structure to 4th order. This recursive structure is the main point of this paper.

We have to note that Conjecture 3.1 is highly nontrivial conjecture, while Conjecture 4.1 is just an algebraic one. In [6], we proposed a scenario of a proof of Conjecture 3.1. However, there are missing modes of perturbation in this scenario which called “zero modes” and we also proposed “zero-mode problem”. The recursive structure in this paper is entirely based on Conjecture 3.1. Therefore, we have to say that “zero-mode problem” is also essential to the recursive structure in the construction of gauge-invariant variables for any-order perturbations.

Here, we discuss the correspondence with the recent proposal of the fully non-linear and exact perturbations by Hwang and Noh [12]. Since we can decompose the n th-order metric perturbation as equation (72), the full metric (10), which is pulled back to \mathcal{M}_0 through a gauge \mathcal{X} , is given by

$$\begin{aligned} \mathcal{X}_\lambda^* \bar{g}_{ab} = & g_{ab} + \sum_{n=1}^k \frac{\lambda^n}{n!} {}^{(n)}\mathcal{H}_{ab} \\ & - \sum_{n=1}^k \frac{\lambda^n}{n!} \sum_{l=1}^n \frac{n!}{(n-l)!} \sum_{\{j_i\} \in J_l} \mathcal{C}_l(\{j_i\}) \mathcal{L}_{-(1)X}^{j_1} \cdots \mathcal{L}_{-(l)X}^{j_l} {}^{(n-l)}\mathcal{X} g_{ab} \\ & + O(\lambda^{k+1}). \end{aligned} \quad (83)$$

Here, in this equation, the term $\sum_{n=1}^k \frac{\lambda^n}{n!} {}^{(n)}\mathcal{H}_{ab}$ is the gauge-invariant part and the second line is the gauge-variant part up to $k + 1$ order. If the right-hand side of equation (83) converges in the limit $k \rightarrow \infty$, the limit $\lim_{k \rightarrow \infty} \sum_{n=1}^k \frac{\lambda^n}{n!} {}^{(n)}\mathcal{H}_{ab}$ corresponds to the gauge-invariant variables in the fully non-linear and exact perturbations proposed by Hwang and Noh [12]. The gauge issue of the fully non-linear and exact perturbations will be

justified in this way.

In the case of cosmological perturbations discussed in Sec. 5, the components of the gauge-invariant part $\lim_{k \rightarrow \infty} \sum_{n=1}^k \frac{\lambda^n}{n!} {}^{(n)}\mathcal{H}_{ab}$ for the fully non-linear and exact perturbations are given by

$$\begin{aligned} \lim_{k \rightarrow \infty} \sum_{n=1}^k \frac{\lambda^n}{n!} {}^{(n)}\mathcal{H}_{ab} = a^2 \{ & -2^{(f)}\Phi(d\eta)_a(d\eta)_b + 2^{(f)}\nu_p(d\eta)_{(a}(dx^p)_{b)} \\ & + (-2^{(f)}\Psi\gamma_{pq} + {}^{(f)}\chi_{pq})(dx^p)_a(dx^q)_b \}, \end{aligned} \quad (84)$$

where

$${}^{(f)}\Phi := \lim_{k \rightarrow \infty} \sum_{n=1}^k \frac{\lambda^n}{n!} {}^{(n)}\Phi, \quad (85)$$

$${}^{(f)}\nu_p := \lim_{k \rightarrow \infty} \sum_{n=1}^k \frac{\lambda^n}{n!} {}^{(n)}\nu_p, \quad (86)$$

$${}^{(f)}\Psi := \lim_{k \rightarrow \infty} \sum_{n=1}^k \frac{\lambda^n}{n!} {}^{(n)}\Psi, \quad (87)$$

$${}^{(f)}\chi_{pq} := \lim_{k \rightarrow \infty} \sum_{n=1}^k \frac{\lambda^n}{n!} {}^{(n)}\chi_{pq}. \quad (88)$$

However, we have to keep in our mind the fact that we ignored “zero modes” to define the variable ${}^{(n)}\Phi$, ${}^{(n)}\nu_p$, ${}^{(n)}\Psi$, and ${}^{(n)}\chi_{pq}$.

Finally, we have to emphasize that the ingredients of this paper are also purely kinematical, since the issue of gauge dependence is purely kinematical. Actually, we do not use any information of field equations such as the Einstein equation. Therefore, the ingredients of this paper are applicable to any theory of gravity with general covariance.

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Appendix A. Properties of the set J_l

In [10], Sonogo and Bruni introduced the set of integer J_l associated with the integer $l \geq 1$ defined by

$$\begin{aligned} J_l &:= \left\{ (j_1, \dots, j_n, \dots) \left| j_n \in \mathbb{N}, \quad \sum_{i=1}^{\infty} i j_i = l \right. \right\} \\ &=: {}_1J_l, \end{aligned} \quad (A.1)$$

where \mathbb{N} is the set of natural numbers. Here, it is convenient to introduce the set J_0 so that

$$J_0 := \{(j_1, \dots, j_n, \dots) \mid j_n = 0 \quad \forall n \in \mathbb{N}\} \quad (\text{A.2})$$

Due to this introduction J_0 , we may regard the definition (A.1) of J_l for $l \geq 0$.

To classify the elements of J_l , we first introduce the set

$${}_1J_l^+ := \{(j_1 + 1, j_2, \dots) \mid (j_1, \dots, j_l, \dots) \in {}_1J_l\}. \quad (\text{A.3})$$

We note that

$${}_1J_0^+ = \{(1, 0, 0, \dots)\} = {}_1J_1. \quad (\text{A.4})$$

If we replace $j_1 \rightarrow j_1 + 1$ in the condition $\sum_{i=1}^{\infty} ij_i = l$ of the definition (A.1), we obtain

$$j_1 + \sum_{i=2}^{\infty} ij_i = l - 1. \quad (\text{A.5})$$

Therefore, ${}_1J_{l-1}^+$ is a subset ${}_1J_l$, namely, the elements of ${}_1J_{l-1}^+$ is the elements of ${}_1J_l$ with $j_1 \geq 1$. All elements of the set ${}_1J_l \setminus {}_1J_{l-1}^+$ have the property $j_1 = 0$.

Second, we consider the set ${}_1J_l \setminus {}_1J_{l-1}^+$. We define ${}_2J_l^+$ by

$${}_2J_l^+ := \{(j_1, j_2 + 1, j_3, \dots) \mid (j_1, j_2, j_3, \dots) \in {}_1J_l \setminus {}_1J_{l-1}^+\}. \quad (\text{A.6})$$

Since all elements in the set ${}_1J_l \setminus {}_1J_{l-1}^+$ have the property $j_1 = 0$, all elements in the set ${}_2J_l^+$ also have the property $j_1 = 0$. Furthermore, since the elements in ${}_1J_l \setminus {}_1J_{l-1}^+$ satisfy the condition $\sum_{i=2}^{\infty} ij_i = l$, the elements of the set ${}_2J_l^+$ satisfy the property $\sum_{i=2}^{\infty} ij_i = l + 2$.

This implies that the set ${}_2J_{l-2}^+$ is the subset of the set ${}_1J_l \setminus {}_1J_{l-1}^+$ with the property $j_2 \geq 1$. We note that all elements of the set ${}_1J_l \setminus ({}_1J_{l-1}^+ \oplus {}_2J_{l-2}^+)$ have the property $j_1 = j_2 = 0$. We also note that ${}_2J_1^+$ is an empty set.

Similarly, we consider the set ${}_1J_l \setminus ({}_1J_{l-1}^+ \oplus {}_2J_{l-2}^+)$. We also define ${}_3J_l^+$ by

$${}_3J_l^+ := \{(j_1, j_2, j_3 + 1, j_4, \dots) \mid (j_1, j_2, j_3, \dots) \in {}_1J_l \setminus ({}_1J_{l-1}^+ \oplus {}_2J_{l-2}^+)\}. \quad (\text{A.7})$$

Since all elements in the set ${}_1J_l \setminus ({}_1J_{l-1}^+ \oplus {}_2J_{l-2}^+)$ have the property $j_1 = j_2 = 0$, all elements in the set ${}_3J_l^+$ also have the property $j_1 = j_2 = 0$. Furthermore, since the elements in ${}_1J_l \setminus ({}_1J_{l-1}^+ \oplus {}_2J_{l-2}^+)$ satisfy the condition $\sum_{i=3}^{\infty} ij_i = l$, the elements of the set

${}_3J_l^+$ satisfy the property $\sum_{i=2}^{\infty} ij_i = l + 3$. This implies that the set ${}_3J_{l-3}^+$ is the subset of the set ${}_1J_l \setminus ({}_1J_{l-1}^+ \oplus {}_2J_{l-2}^+)$ with the property $j_3 \geq 1$. We note that all elements of the set ${}_1J_l \setminus ({}_1J_{l-1}^+ \oplus {}_2J_{l-2}^+ \oplus {}_3J_{l-3}^+)$ have the property $j_1 = j_2 = j_3 = 0$ and the sets ${}_3J_l^+$ with $l = 1, 2$ are empty sets.

We can repeat this classification of the elements in ${}_1J_l$ through the recursive definitions of the sets

$${}_k J_l^+ := \left\{ (j_1, \dots, j_{k-1}, j_k + 1, j_{k+1}, \dots) \mid (j_1, \dots, j_k, \dots) \in {}_1J_l \setminus \left(\bigoplus_{p=1}^k {}_p J_{l-p}^+ \right) \right\},$$

$$(A.8)$$

for $0 \leq k \leq l$. This classification of the elements in ${}_1J_l$ terminates when $k = l$ and we obtain the results

$$J_l =: {}_1J_l = \bigoplus_{k=1}^l {}_k J_{l-k}^+. \quad (A.9)$$

We note that

$${}_k J_{l-k}^+ = \emptyset \quad \text{for} \quad k > l - k > 0. \quad (A.10)$$

and

$$J_0^+ = \{(0, \dots, 0, j_l = 1, 0, \dots)\}. \quad (A.11)$$

The explicit elements of ${}_1J_1$, ${}_1J_2$, ${}_1J_3$, and ${}_1J_4$ are given by

$${}_1J_1 = \{(1, 0, 0, 0, 0, 0, \dots)\}, \quad (A.12)$$

$${}_1J_2 = \{(2, 0, 0, 0, 0, 0, \dots), \\ (0, 1, 0, 0, 0, 0, \dots)\}, \quad (A.13)$$

$${}_1J_3 = \{(3, 0, 0, 0, 0, 0, \dots), \\ (1, 1, 0, 0, 0, 0, \dots), \\ (0, 0, 1, 0, 0, 0, \dots)\}, \quad (A.14)$$

$${}_1J_4 = \{(4, 0, 0, 0, 0, 0, \dots), \\ (2, 1, 0, 0, 0, 0, \dots), \\ (1, 0, 1, 0, 0, 0, \dots), \\ (0, 2, 0, 0, 0, 0, \dots), \\ (0, 0, 0, 1, 0, 0, \dots)\}. \quad (A.15)$$

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